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History and Prehistory of the Riccati Equation

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1. Introduction

On a cold New Year Eve of 1720, Count Jacopo Francesco Riccati, a nobleman living in the Republic of Venice, wrote a letter to his friend Giovanni Rizzetti, where he proposed two new differential equations. In modern symbols, these equations can be written as follows:

$$\dot{x} = \alpha x^2 + \beta t^m \quad (1)$$

$$\dot{x} = \alpha x^2 + \beta t + \gamma t^2 \quad (2)$$

where m is a constant and t is the independent variable. This is probably the first document witnessing the early days of the Riccati Equation, an equation which was to become of paramount importance in our days.

2. Riccati’s Life at a Glance

Born in Venice on May 28, 1676, Riccati lost his father at the age of 10 only. His education took place in a Jesuit’s College in Brescia, where he enrolled in 1687, probably with no intention of ever becoming a scientist. Indeed, at the end of the college days, in 1693, he began to study law at the University of Padua. However, following his natural inclination, he also attended classes of astronomy given by Father Stefano degli Angeli, a former pupil of Bonaventura Cavalieri. Father Stefano was fond of Isaac *Newton’s Philosophiae Naturalis Principia*, which he passed on to young Riccati around 1695. This is probably the event which caused Riccati to turn from law to science.

After graduating on June 7, 1696, he married Elisabetta dei Conti d’Onigo on October 15, 1696. She bore him 18 children, of whom 9 survived childhood. Among them, Vincenzo (b.1707, d.1775), a mathematical physicist, and Giordano (b.1709, d. 1790), a scholar with many talents, but with a special interest for architecture and music, are worth mentioning.

Riccati spent most of his life in Castelfranco Veneto, a little town located in the beautiful country region surrounding Venice. Besides taking care of his family and his large estate, he was in charge of the administration of Castelfranco Veneto, as Provveditore (Mayor) of the town, for nine years during the period 1698-1729. He also owned a house in the nearby town of Treviso, where he moved after the death of his wife (1749), and where his children had been used to spending a good part of each year after 1747.

Notwithstanding all his responsabilities, Count Riccati always found time for his beloved studies. He did not follow any lecture courses in mathematics or other scientific disciplines. Basically, the profound knowledge of the self-thought man was acquired by reading and exchange of ideas with other scientists, through correspondence and conversation. Riccati was in contact with Domenica Maria Gaetani Agnesi, Gabriele Manfredi, Giovanni Poleni, Giovanni Rizzetti, Giuseppe Suzzi, Antonio Vallisneri, Bernardino Zendrini, and many other left-brained Italians. He was also in contact with various European mathematicians, such as Jacob Hermann and some members of the influential Bernoulli family, mainly Nicolaus III (b1695, d.1726). Most of Riccati's correspondence can be found in Castelfranco Veneto, with the exception of the letters exchanged with members of the Bernoulli family, which are kept in the Basel University Library.

Riccati was an undemonstrative, kind man who preferred his home to academies and universities. His way of life was a very simple one, and he travelled very little. He turned down many notable invitations, including the most appealing one of becoming president of St. Petersburg's Academy (c. 1725). He also refused the chair of Mathematics at the University of Padua and the invitation to the Court of Wien as Adviser.

Count Riccati was a strong and hard-working person, with an active and fertile mind throughout the years of his life. On April 2, 1754, he had a sudden bout of fever and a fortnight later, on April 15, he passed away.

3. Riccati and Differential Equations

An appropriate way of appreciating Riccati's contribution to differential calculus is to consult his *Opere*, a work in 4 volumes edited by his son Giordano and published in Lucca by G. Rocchi in 1765, after Riccati's death. However, if one is willing to follow the true sequence of Riccati's discoveries over the years, one should complement the reading of his publications with the correspondence he wrote and received.

Riccati's main interest in the area of differential equations focused on the methods of separation of variables. Probably, such an interest originated in the reading of Gabriele Manfredi's book *De constructione aequationum differentialium primi gradus* printed in Bologna in 1707 (Manfredi occupied the Chair of Mathematics at Bologna University for many years).

Originally, Riccati attention focused on the following problem of geometric type: suppose that a point of coordinates (α, β) describes a trajectory in the plane according to a linear differential equation of the first order, i. e.:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \end{bmatrix} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix}$$

The question is: which is the equation governing the slope $x = \beta / \alpha$?

It is easily seen that the equation is

$$\dot{x} = ax^2 + bx + c$$

where

$a = -w_{12}$

$b = w_{22} - w_{11}$

$c = w_{21}$.

This equation, characterised by a quadratic expression at the right hand side, is exactly the equation on which Riccati's interest evolved around. He dealt with equations with constant and time-varying parameters, with special attention devoted to (1), (2) and

$\dot{x} = \alpha t^p x^2 + \beta t^m$ (3)

$\dot{x} = \alpha t^p x^q + \beta t^n$. (4)

Count Riccati developed various methods for the determination of the solution of these equations, such as the method of *dimezzata separazione* (about 1715), and of *coefficienti ed esponenti indeterminati* (about 1717).

A compendium of Riccati's methods can be found in the lecture notes which he prepared for his private classes to Giuseppe Suzzi and Ludovico da Riva, who studied mathematics with him during 1722 and 1723. Subsequently, Suzzi and da Riva became professors of, respectively, physics and astronomy at the University of Padua. The lecture notes, which can be found in the Opera, are entitled *Della separazione delle indeterminate nelle equazioni differenziali di primo e di secondo grado, e della riduzione delle equazioni del secondo grado e d'altri gradi superiori* (on the separation of variables in differential equations of first and second order, and on the reduction of differential equations of second order and higher orders). The notes (154 pages) are comprised of three parts and two appendices. In the first part (*Dei metodi inventati dall'autore per separare le indeterminate nelle equazioni differenziali di primo grado*), the methods of solution due to Riccati are discussed with reference to different equations which we would now call " Riccati equations ".

4. The Riccati Equation in Calculus of Variations, Optimal Filtering and Control, and Other Problems

Once Riccati days had passed, his equation has been studied by many, in particular Euler (1760 circa) and Jacques Liouville (1840 circa). However, it is in the 20th century that the equation reaches a paramount importance, especially for the developments of Calculus of Variations and Optimal Filtering and Control.

The roots of Calculus of Variations go back to Fermat's principle of least time (1662). However, the problem which gave the development of the field its first momentum was the celebrated *brachystochrone problem* stated by Galileo's (1638) and solved by John Bernoulli's (1697). The brachystochrone problem can be stated as follows. Consider a curve in a vertical plane, starting at a given point *a* and ending at another given point *b*, located at a lower ordinate. A point mass slides along such a curve, under the effect of gravity, so moving from *a* to *b* in some interval of time (transfer time). Which is the curve connecting *a* to *b* resulting in the shortest transfer time?

Following the successful solution of the brachystochrone problem, many other variational problem were studied in the subsequent centuries, from the search of geodesic lines to isoperimetric problems, from nautical paths in stationary sea currents to the Zermelo navigation problem. We refer the interested reader to the book by Caratheodory, *autore degno di fede e di ubbidienza*, a milestone in the area.

However, it is mainly with the development of *second variations methods* that the Riccati equation enters the field, mainly in the early decades of the 20th century. One of the basic formulations of a variational problems is the following one. Consider the functional

$$J[x] = \int_a^b F\left(t, x, \dot{x}\right) \quad dt$$

where *F* is a smooth function, over the class of curves $x = x(t)$ with fixed end points:

$$x(a) = A \qquad ; \qquad x(b) = B\,.$$

The objective of finding conditions for the characterisation of the solution is pursued by giving the function *x(t)* an increment *h*(*t*) such that

$$h(a) = 0 \qquad ; \qquad h(b) = 0\,.$$

A Taylor expansion of the *F* leads to

$$\Delta J[h] = J[x+h] - J[x] =$$

$$= \delta J[h] + \delta^2 J[h] + \varepsilon$$

where

$$\delta J[h] = \int_a^b \left(F_x h + F_{\dot{x}} \dot{h}\right) dt$$

is the so-called *first variation*

$$\delta^2 J[h] = \frac{1}{2}\int_a^b \left(F_{x\ x} h^2 + 2F_{x\ x} h\dot{h} + F_{x\ x}\dot{h}^2\right) dt$$ is the *second variation*, and ε is an infinitesimal of higher order. By integrating by parts, one obtains

$$\delta^2 J[h] = \int_a^b \left(P\dot{h} + Qh^2\right) \quad dt \qquad (5)$$

where

$$P = P(t) = \frac{1}{2} F_{x\ x}$$

$$Q = Q(t) = \frac{1}{2}\left(F_{x\ x} - \frac{d}{dx} F_{x\ \dot{x}}\right).$$

It can be shown that a necessary condition for a functional $J[x]$ to have a minimum is that its second variation $\delta^2 J[h]$ be nonnegative. This opened the road to a vast literature devoted to the study of the sign of the above expression (5) for $\delta^2 J[h]$. Now, in (5) the term $P\dot{h}^2$ plays a dominant role, as is easily seen. Therefore, it can be argued that the condition $P(t) \geq 0 \quad \left(i.e.\ F_{\dot{x}\ \dot{x}}\right) \geq 0$ is necessary in order for $x(t)$ be the solution of the problem.

A lot of attention has also been paid by many outstanding scientists to the issue whether the condition $F_{\dot{x}\ \dot{x}} > 0$ is sufficient for a weak minimum. The main reason behind this conjecture can be explained according to the following considerations, originating in Legendre's work. Take an arbitrary function *w*(*t*) and note that, since *h*(*a*) = *h*(*b*) = 0,

$$\int_a^b \frac{d}{dt}\left(wh^2\right)dt = \int_a^b \left(\dot{w}h^2 + 2wh\dot{h}\right)dt = 0$$

Therefore:

$$\delta^2 J[h] = \int_a^b \left[P\dot{h}^2 + 2wh\dot{h} + \left(Q + \dot{w}\right)h^2\right]dt \qquad (6)$$

Now, suppose that *w*(*t*) satisfies the first order differential equation

$$\dot{w} = \frac{w^2}{P} - Q \quad . (7)$$

Then, the integrand of (6) would take the perfect square expression

$$\delta^2 J[h] = \int_a^b \frac{\left(P\dot{h} + wh\right)^2}{P}dt \quad ,$$

and the condition *P*(t)> 0 would suffice.

As Legendre knew, however, the perfect square trick does not entirely work, since eq. (7) may not admit a solution over the whole interval [a,b] . This issue was subsequently clarified by Lagrange and Jacobi, not to speak of Hamilton, who showed that the condition $F_{\dot{x}\ \dot{x}} > 0$ alone is not sufficient in general, and introduced a more elaborated condition involving the concept of conjugate points.

Now, our reader certainly recognized in (7) a *scalar Riccati equation*. If a more general performance index with *n* unknown $x_1(t), x_2(t), \ldots\ldots, x_n(t)$ were considered, then the Matrix Riccati equation:

$$\dot{W} = WP^{-1}W - Q$$

would enter the scenario, where *W* is a symmetric matrix of smooth functions of dimension *nxn*.

While the literature on the second variation methods in Calculus of Variations mainly developed in the first half of the 20th century, Optimal Filtering and Control entered the scientific stage with Kalman's contribution mainly during the decade 1960-1970, and stimulated the research activity for the remaining portion of the century. The problem dealt with in this new field is also a functional optimizitation problem, but, a new challenging issue is there; the presence of *exogenous variables*, affecting the dynamics of a phenomenon described in *state-space* form. Under the impetus of Kalman work, such a modelization of the real world had to become of paramount importance for diverse fields of investigation and engineering application for the years to come. Thanks to such a unifying model, phenomena, plants, devices, processes did not need any more a conceptual diversification. They just became "*a System*", and all the physical background behind faded out, its role being confined to the preliminary modelling phase only. Another characteristic of Kalman modelization is the possibility it offers of easily incorporating the effect of disturbances, a major ingredient for the formidable communication and information problems of the 20th century. Finally, we should note that, while the independent variable in Calculus of Variations is typically thought of as a *space* coordinate, the typical independent variable of Optimal Filtering and Control is *time*.

Under state-space representations, the optimality conditions are naturally posed in terms of Riccati equations. One of the paradigm of optimal control, the so called *Linear Quadratic Gaussian* (LQG) problem, consists in the optimisation of a quadratic cost function associated with a linear dynamical systems, described - as already said - in state-space form, and subject to additive disturbances, seen as white and Gaussian noises. This problem calls for the solution of two matrix Riccati equations, one associated with the filtering side of the problem, and a dual one, associated with the control side.

The original Riccati interest was of course for *differential* equations. Also the Calculus of Variations was essentially a matter of continuous-time equations. On the contrary, in Optimal Filtering and Control there is a notable interest for the difference version of the optimal problem, leading to the discrete-time version of the Riccati equation, also universally named after Riccati.

The solutions of optimal filtering and control problems, are in general function of the independent variable. In many situations, however, the solutions of the Riccati equations converge as *t* diverges. The asymptotic convergence point is the solution of an algebraic equation, the so-called *algebraic Riccati equation*. A lot of attention has been paid to such an equation, since it is the basis for a time invariant (suboptimal) solution.

Differential, difference, and algebraic Riccati equations have been extensively studied in the systems and control literature, and their properties are now well understood, as discussed in the papers and volumes on the subject.

We end this conversation, by mentioning that the Riccati equation plays a signficant role in other diverse applications, which cannot be reported here due to lack of space. Some of them are discussed in the volume by W.T. Reid quoted below.

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