Stochastic Identification and Digital Control of a Heat Exchanger: a Simulation Test Case†

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ABSTRACT: In this paper, an extended simulation study is presented by which the power of modern digital identification and control methodology is tested against the nontrivial dynamics of a relatively simple yet nonelementary plant. Many critical issues which may significantly affect the success of the entire design process can be discussed along the lines of this study. The plant considered here is a liquid-saturated steam heat exchanger, where the liquid flow-rate is manipulated to regulate the outlet liquid temperature. All the numerical experiments have been conducted by digital simulation of a fairly standard nonlinear distributed parameter model.

I. Introduction

In the last few years, stochastic identification and digital control theory have come to a certain maturity as noted by the recent appearance of several books on the subject, see (1–6).

The purpose of this paper is to test the applicability and limitation of the many facets of this theory by an extended simulation study. The plant considered is a common liquid–saturated steam heat exchanger, where the flow-rate is manipulated in order to regulate the water outlet temperature in spite of disturbances acting on the steam and water inlet temperatures.

Heat exchanger dynamics have been extensively considered in process control literature. See, for instance, (7–10). However, stochastic identification and digital control experiments for this kind of processes are reported in a few papers only, e.g. (11), (5, Ch. 30) and (12). A non-standard approach to the tuning regulation is presented in (13). The approach used in this paper consists of the application of the following three steps: stochastic identification, digital control and self-tuning control.

Stochastic identification. The dependence of the water outlet temperature upon the flow-rate is described by an ARMAX model, namely by a discrete-time, stochastic, linear, time-invariant and finite-dimensional model. The process considered being nonlinear and infinite dimensional, the identification of such a model is indeed nontrivial. Among other things, it requires a sensible planning of the identification experiment and calls for an extensive use of model selection criteria, so

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as to achieve a suitable compromise between accuracy and complexity. These issues are central to the present setting of stochastic identification. For a tutorial introduction to this topic, see (14).

**Digital control.** The dynamics of the identified model is characterized by a number of zeros quite near the instability region. Therefore, cancellation controllers, which form the basis of many digital control schemes [see e.g. (5, Ch. 6)] cannot be applied. Instead, we resort to the so-called generalized minimum-variance control approach, (15, 16), which can be successfully implemented to handle these situations.

**Self-tuning control.** Many adaptive control schemes have been discussed in the literature. For instance, more than 500 references can be found in Landau's book (2). The approach adopted here is the self-tuning version of the generalized minimum-variance control design mentioned before. The results previously obtained by means of the known-parameter controllers act here as a standard against which the responses of the automatically tuned controllers can be measured.

The results presented in this paper summarize the stochastic aspects of a project developed in the last few years at the Politecnico di Milano (Italy), mainly by the students in Systems and Control Engineering. The project was conceived for tutorial purposes, with the objective to assess the reliability of stochastic digital control techniques by comparing the control performances obtained with the ones achievable by classical design methods. The same objective applies to this paper as well. For reasons of brevity, however, the part concerning classical controllers is here omitted. Furthermore, the various methods will be described as concisely as possible, referring the interested reader to the appropriate bibliography.

The paper is organized as follows. After the introduction of some notational conventions (Section II), we deal with the process description and the problem statement in Section III. The identification procedure is subsequently described in Sections IV–VI. Basically, it consists of three parts. First, a suitable identification experiment is designed. Then, the parameters of a number of ARMAX models of different complexity are estimated using the extended least-squares algorithm (17, 6). The model finally adopted is selected on the basis of both “objective” and “subjective” criteria. As for the former class of criteria, reference is made to the Akaike and Rissanen criteria (18, 19). As for the latter, the whiteness of the model prediction error is tested. The concept of generalized minimum-variance control is the subject of Section VII. A number of different design techniques fall under this heading. Among them, the so-called detuned model reference technique, see e.g. (20), is applied here. The control systems thus obtained are checked against a series of step changes in the principal variables (Section VIII). Finally, the self-tuning version of this technique is briefly outlined in Section IX and then applied to the heat exchanger control in Section X.

### II. Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$R^n$</td>
<td>space of real $n$-dimensional vectors</td>
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<tr>
<td>$t$</td>
<td>continuous time ($t \in R^1$)</td>
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<tr>
<td>$k$</td>
<td>discrete time ($k = 0, 1, 2, ...$)</td>
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<tr>
<td>$z^{-1}$</td>
<td>unit delay operator ($z^{-1}y(k) = y(k - 1)$)</td>
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<tr>
<td>$\dot{A}$</td>
<td>derivative of a matrix $A$ w.r.t. $t$</td>
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\( A_x \) \( \quad \text{derivative of } A \text{ w.r.t. } x \)

\( \text{IN}(0, \sigma^2) \) \( \quad \text{Gaussian white noise with zero mean value and variance } \sigma^2 \)

**Heat exchanger symbols (see also Fig. 1; the integer } j = 1, 2, \text{ refers to liquid and metal, respectively)**

- \( l \) \( \quad \text{tube length} \)
- \( D_l, D_s \) \( \quad \text{diameters} \)
- \( \theta_j(t, x) \) \( j = 1, 2, \text{ temperatures} \)
- \( c_j \) \( j = 1, 2, \text{ specific heats} \)
- \( \mu_j \) \( j = 1, 2, \text{ linear densities} \)
- \( \bar{\theta}_j(t) \) \( \text{inlet liquid temperature } (= \theta_1(t, 0)) \)
- \( \bar{\theta}_o(t) \) \( \text{outlet liquid temperature } (= \theta_1(t, L)) \)
- \( \theta_s(t) \) \( \text{steam temperature} \)
- \( q(t) \) \( \text{flow rate} \)
- \( v(t) \) \( \text{liquid velocity} \)
- \( \alpha_l \) \( \text{coefficient of heat transfer liquid/metal} \)
- \( \alpha_s \) \( \text{coefficient of heat transfer steam/metal} \)

A bar over a symbol denotes nominal value. Thus \( \bar{\theta}_{t_0}, \bar{\theta}_{t_0}, \bar{\theta}_s, \bar{\theta}_x, \bar{\theta}_z \) are the nominal values of \( \hat{\theta}_{t_0}, \hat{\theta}_{t_0}, \theta_s, \theta_x, \theta_z \), respectively.

### III. Process Description and Problem Statement

In the heat exchanger, water is made to circulate once through a copper tube, saturated superheated steam flowing on the outside. Assuming that there are no radial temperature gradients, the temperatures \( \theta_1 \) and \( \theta_2 \) of the liquid and the metal are functions only of time \( t \) and axial coordinate \( x : \theta_1 = \theta_1(t, x), \theta_2 = \theta_2(t, x) \).

The steam being saturated, its temperature \( \theta_x \) is a function of time only: \( \theta_x = \theta_x(t) \).

Energy balances in both liquid and metal sections of the exchanger lead to the following partial differential equations:

\[
\begin{align*}
\tau_1 \dot{\theta}_1(t, x) + \tau_1 v(t) \theta_2(t, x) + \theta_1(t, x) - \theta_2(t, x) &= 0 \\
\tau_2 \dot{\theta}_2(t, x) + \beta(\theta_2(t, x) - \theta_1(t, x)) + \theta_2(t, x) - \theta_x(t) &= 0
\end{align*}
\]

where

\[
\begin{align*}
\tau_1 &= \frac{\mu_1 c_1}{\alpha_l \pi D_l} \\
\tau_2 &= \frac{\mu_2 c_2}{\alpha_s \pi D_s} \\
\beta &= \frac{\alpha_l D_l}{\alpha_s D_s}
\end{align*}
\]

The steam/metal heat transfer coefficient \( \alpha_s \) may be assumed to be a constant. On the contrary, the liquid/metal heat transfer coefficient \( \alpha_l \) increases with the velocity of the liquid. Such a dependence is given by the Dittus-Boelter relation (9), namely

\[
\alpha_l = \alpha_l(t) = kv(t)^v, \quad v = 0.8,
\]
where \( k \) is a constant. Consequently, denoting the nominal values of \( v, \tau \) and \( \beta \) by \( \bar{v}, \bar{\tau} \) and \( \bar{\beta} \) respectively, it follows from (3) and (4) that

\[
\tau_1 = \bar{\tau}_1 \left( \frac{\bar{v}}{v(t)} \right)^\nu, \quad \beta = \bar{\beta} \left( \frac{v(t)}{\bar{v}} \right)^\nu.
\]

Denoting by \( L \) the length of the tube, let \( \tilde{\theta}_{il}(t) = \theta_{i}(t, 0) \) and \( \tilde{\theta}_{io}(t) = \theta_{i}(t, L) \) be the inlet and outlet liquid temperatures. The problem is to control the outlet liquid temperature by adjusting the liquid flow-rate. The process output variable is measured by a transducer approximately described as a first-order system whose output is denoted by \( \theta_{io} \)

\[
\dot{\theta}_{io}(t) = \gamma \theta_{io}(t) + \delta \tilde{\theta}_{io}(t).
\]

For digital control purposes, the variable \( \theta_{io} \) is sampled at uniform sampling rate giving rise to a signal

\[
y(k) = \theta_{io}(kA - \varepsilon) - \tilde{\theta}_{io}
\]

where \( \tilde{\theta}_{io} \) is the nominal value of \( \theta_{io} \) and \( \varepsilon \ll \Delta \) is introduced to allow for the computation time of the controller, see e.g. (21). As for the velocity, it is determined over time interval \( t \in [k\Delta, (k+1)\Delta] \) by a discrete time signal \( u(k) \) as follows.

\[
v(t) = \begin{cases} v_m, & \text{if } \bar{v} + u(k) \leq v_m \\ \bar{v} + u(k), & \text{if } v_m < \bar{v} + u(k) < v_M \\ v_M, & \text{if } \bar{v} + u(k) \geq v_M \end{cases}
\]

Saturations \( v_m \) and \( v_M \) are introduced to model the valve nonlinearity. The choice of sampling interval \( \Delta \) will be discussed in Section IV as part of the identification experiment design.

The block diagram of the process is given in Fig. 2, while both data and nominal values are specified in Appendix 1. The process was simulated on a PDP11/34 computer. The partial differential equations, (1) and (2), have been integrated along the characteristics, using a predictor-corrector method (22). For conciseness, we will not dwell upon the simulation aspects of the heat exchanger here. The reader is referred to (23) where every detail on the simulation of the heat exchanger can be

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**Fig. 1.** The heat exchanger schematization.
found. However, the following remarks point out a couple of relevant features of the exchanger.

**Remark 1**

A number of responses to step variations in the main variables about the nominal point are reported in Fig. 3. It can be seen in Fig. 3(a) that there is a delay of about 8 s in the responses to the inlet liquid temperature variations. In general, such a delay results from a combination of two effects: on the one hand, there is the pure delay due to transportation along the tube and, on the other, that produced by the heating "inertia" of the tube. The latter is known as the "percolation effect" (7, 8). Of the two heat transfer coefficients, the steam/metal one is much greater than the other coefficient (see Appendix 1). It is therefore reasonable to neglect any temperature difference between the tube and the steam components of the system. This is confirmed by the results of Fig. 3(a), which show that the experimental delay in the step response does coincide with the transportation delay.

Figure 3(b) refers to the step response of outlet temperature to variations in velocity. Note, in particular, the 90% settling time, which will play a major role in the identification experiment design discussed in the next section.

**Remark 2**

The following transfer function, relating variations in measured outlet liquid...
temperature to variations in velocity, can be derived by linearizing (1) and (2):

\[
W(s) = \frac{6.181(1 + 0.158s)(1 + 6.043s)}{s(1 + s)(1 + 0.26s)(1 + 5.325s)}
\times \left[ 0.389 - e^{-0.944 \frac{(1 + 4.665s)(1 + 11.123s)}{(1 + 0.043s)}} \right].
\]

Figure 4 shows the amplitude and phase diagrams of \( W \). The peaks appearing in both diagrams are peculiar to heat exchangers dynamics (7). These Bode plots are useful to interpret some identification and self-tuning results.

![Figure 4](image.png)

**Fig. 4.** Process and model frequency responses: the continuous lines represent the diagrams of transfer function \( W(s) \) obtained by linearizing the process equations about the nominal point; the dotted lines represent the diagrams of the identified model transfer function \( T(z) \).
IV. Identification Experiment Design

Any identification experiment must be planned according to the intended use of the model. Keeping in mind the problem statement of Section III the purpose of the identification procedure is to derive a discrete time linear dynamical model relating the outlet temperature to the liquid velocity about the nominal value. Therefore, the experiment is designed as follows.

Input signal design

The system of Fig. 2 has three inputs, namely, variations in fluid velocity, fluid inlet temperature and steam temperature. For the sake of simplicity, the latter variables are kept constant and equal to their nominal value, \( \theta_{\text{in}}(t) = \bar{\theta}_{\text{in}} \) and \( \theta_{\text{st}}(t) = \bar{\theta}_{\text{st}} \), in all the performed experiments. As for the fluid velocity variation, it has been modeled as a Gaussian white noise, \( u \sim \mathcal{N}(0, \lambda^2) \).

This choice was made in view of its simplicity in simulation as well as its persistent excitation properties (24). The noise variance \( \lambda^2 \) has been chosen as the largest value for which a linear model of the process about the nominal point still makes sense. A number of responses of the outlet liquid temperature to step changes in the fluid velocity were analyzed. All the step responses have diagrams which agree qualitatively with that depicted in Fig. 3(b). However, due to the process nonlinearity, the 90% settling time and the steady state response gain do depend upon the magnitude of the velocity variation. Such a dependence is pointed out in Fig. 5, where \( \bar{\mu} \) and \( T_s \) correspond to the intersections of the two diagrams with the vertical axis.

![Characteristics of the exchanger response to a step change in the liquid velocity of amplitude \( A \) (\( \bar{\mu} = 21.8^\circ \text{C m}^{-1}, T_s = 8.4 \text{ s} \)). The amplitude \( A \) is in m s\(^{-1} \).](image)
It can be seen from this diagram that the maximum amplitude of the liquid velocity step change for which both \( T_a \in [0.9T_a, 1.1T_a] \) and \( \mu \in [0.9\mu, 1.1\mu] \) is approximately. This value can be tentatively interpreted as an upper bound for the velocity variation resulting in a process behaviour that can be safely assumed to be linear. Since the probability that a Gaussian variable of variance \( \lambda^2 \) belongs to the interval \([-3\lambda, 3\lambda]\) is greater than 99%, the value of \( \lambda \) has been taken equal to 0.01 ms\(^{-1}\). This choice will be subject to a validation test below.

**Sampling interval**

The sampling interval \( \Delta \) has to be adjusted according to the process dynamics. A well known rule of thumb [see e.g. (5)] is that \( \Delta \approx \left( \frac{1}{15} T_a, \frac{1}{T} \right) \). The value \( \Delta = 1 \) s has been chosen, in view of the 90% settling time range (Fig. 5).

**Collected data**

Having made the foregoing choices, the system of Fig. 2 was driven by a Gaussian white noise sample of a standard deviation \( \lambda = 0.01 \) ms\(^{-1}\). The system output was recorded with a sampling rate \( \Delta = 1 \) s. Moreover, \( \varepsilon = 0.05 \) s was chosen in (5), taking into account that 0.05 s was the integration step of (1) and (2).

The identification was based on the data so collected \( u(1) u(2) \ldots u(N) \) \( y(1) y(2) \ldots y(N) \). A large value of \( N \) was taken, \( N = 4000 \), in order to avoid any failures of the identification and control methods due to lack of information.

**Validation of the linearity assumption**

A stable linear system fed by a Gaussian input produces a Gaussian output (25). Therefore the extent to which a linear system can be used to model the heat exchanger dynamics can be checked by testing the Gaussianity of the collected data \( y \). The Gaussianity assumption of \( y \) was verified at the 1% significance level by applying the Kolmogorov–Smirnov test, see (26). However, the assumption has to be rejected at the 5% level. This indicated that the process behaviour is "at the linearity limit" under the experimental conditions. Thus, the choice of the input variance made above seems reasonable.

**V. Identification Procedure**

The class of models considered in this paper in the well known ARMAX family, which is extensively used in identification. A model is said to belong to this family if it is given by the difference equation:

\[
A(z^{-1})y(k) = B(z^{-1})u(k-d) + C(z^{-1})e(k),
\]

where \( e \) is a Gaussian white noise, \( d \) a positive integer (the input delay) and \( A(z^{-1}), B(z^{-1}) \) and \( C(z^{-1}) \) are the following polynomials in the unit delay operator \( z^{-1} \):

\[
A(z^{-1}) = 1 - a_1 z^{-1} - a_2 z^{-2} - \cdots - a_n z^{-n_a}
\]

\[
B(z^{-1}) = b_1 + b_2 z^{-1} + b_3 z^{-2} + \cdots + b_n z^{-n_b+1}
\]

\[
C(z^{-1}) = 1 + c_1 z^{-1} + c_2 z^{-2} + \cdots + c_n z^{-n_c}.
\]
Given $d$, an ARMAX model is specified by the integers $n_a, n_b, n_c$ and the real vector.

$$\xi = |a_1a_2\cdots a_n b_1b_2\cdots b_nb_c c_2\cdots c_n|^t. \quad (9)$$

A subclass of the ARMAX models family is given by the so-called equation error models, which are obtained from (7) by assuming $C(z^{-1}) = 1$, i.e.

$$A(z^{-1})y(k) = B(z^{-1})u(k - d) + e(k). \quad (10)$$

The model parameters vector $\xi$ is then the $n_a + n_b$ vector defined by the parameters of both polynomials $A(z^{-1})$ and $B(z^{-1})$.

A large number of algorithms have been proposed in the econometric and engineering literature to estimate the parameter $\xi$ from input-output records, see (27–30, 6). Due to its simplicity, the least-squares (LS) algorithm, or its recursive version (RLS), is widely used in the estimation procedure. However, except for equation error models, this algorithm leads to biased estimates of parameters $a_i$ and $b_i$. A simple extension of RLS enabling the estimation of the parameters of polynomial $C(z^{-1})$ as well is the so-called extended least-squares (ELS) algorithm. A concise presentation of RLS and ELS is given in Appendix 2. These algorithms will be used in Section VI to fit a number of equation error and ARMAX models to the heat exchanger data.

**Model complexity selection**

Basically, model selection results from a trade-off between complexity and faithfulness. Indeed, other things being equal, the more complex is the model the better it fits the data. Beyond a certain degree of complexity, however, a better fit denotes nothing but an increasing adherence of the model to the noise corrupting the sample at hand (overfit). This problem has been extensively studied, and a number of model complexity selection criteria are available. Among the so-called objective criteria, we may refer to the Akaike and Rissanen criteria (18, 19).

These criteria require minimizing, respectively

$$AIC = \ln \mu^2 + \frac{2}{N} q$$

$$U = \ln \mu^2 + \frac{\ln N}{N} q,$$

where

$$q = n_a + n_b + n_c + 1$$

is the number of free parameters in the model. In turn, $\mu^2$ denotes the variance of the prediction error of the estimated ARMAX model. As for the subjective criteria, an important test is to check the error whiteness, see e.g. (31). Here, the Anderson test [(27), p. 289] has been used. This test is based on counting the number of points $r_s$ of the prediction error correlation function lying outside a confidence interval determined by the level of significance $\alpha$. More precisely, let $s$ be the number of correlation points taken into consideration. Then, $r_s > \alpha s$ results in the whiteness assumption being rejected.
VI. Identification Results

For simplicity, equation error models with $d = 1$ and $n_a = n_b$ have been considered initially. Letting $n_r := n_a = n_b$, the results of the RLS based identification procedure are summarized in Fig. 6(a), where the values of AIC and U for $n = 1, 2, \ldots, 11$ are indicated. As for the Anderson test, the first 30 prediction error correlation points are considered ($s = 30$). The number of points among these which lie outside the 5% confidence interval, denoted by $r_{0.05}$, is depicted in Fig. 7(a). As can be seen, the objective criteria diagrams become flat only for large values of $n$. However, the errors are far from being white even for $n = 11$. Once this has been done, general ARMAX models are considered. Henceforth, the notation $\mathcal{M}_d (n_a, n_b, n_c)$ is used to denote the ARMAX model (7) whose parameters are estimated by the ELS algorithm. The AIC, U and $r_{0.05}$ diagrams relative to a number of models of the type $\mathcal{M}_1 (n, n, n)$ are given in Figs 6(b) and 7(b). For $n \geq 9$, the whiteness hypothesis of the errors is now acceptable. The Akaike and Rissanen criteria are minimized at $n = 10$ and $n = 9$ respectively. Notice also that the difference between the values of U for $n = 9$ and 10 is very slight. Notwithstanding these results, concluding that models $\mathcal{M}_1 (9, 9, 9)$ and $\mathcal{M}_1 (10, 10, 10)$ are good is not easy to accept if one considers that these models are described by 27 or 30 parameters! To obtain simpler good models, the assumption that $n_a = n_b = n_c$ has to be removed. To find out whether $n_a$, $n_b$ or $n_c$ may in fact be reduced, three runs were performed with $n_a = n_b = 10$ and $n_c = 5$, $n_a = n_c = 10$ and $n_b = 5$, $n_a = n_b = 10$ and $n_c = 10$. The results of Table I clearly indicate that what really matters is that $n_b$ is large; $n_a$ and $n_c$ may be smaller though.

To probe further into this point, the models with $n_b = 10$ and $n_a, n_c = 1, 2, \ldots, 10$ were estimated.

According to the Akaike and Rissanen criteria, the best models of this family are $\mathcal{M}_1 (4, 10, 7)$ and $\mathcal{M}_1 (2, 10, 6)$ respectively (Tables II and III). Interestingly enough, the corresponding values of AIC and U are not sensibly worse than the ones associated with models $\mathcal{M}_1 (9, 9, 9)$ or $\mathcal{M}_1 (10, 10, 10)$. Moreover, the prediction errors of $\mathcal{M}_1 (2, 10, 6)$ and $\mathcal{M}_1 (4, 10, 7)$ are apparently white (Table IV).

Many more runs have been performed with $d > 1$ in (7). However, the results obtained clearly show that the models are in that case less accurate. In conclusion,
the identification procedure indicates that models $\mathcal{M}_1(4, 10, 7)$ and $\mathcal{M}_1(2, 10, 6)$ are the best among those considered. An extensive analysis of the models shows that their input–output characteristics can hardly be distinguished. Being more parsimonious, model $\mathcal{M}_1(2, 10, 6)$ has been considered for control purposes. This model is given in Appendix 3. Notice that ARMAX model (7) can also be written as

$$y(k) = T(z)u(k-d) + V(z) e(k),$$

where

$$T(z) = \frac{z^{n_b-1} B(z^{-1})}{z^{n_a-1} A(z^{-1})}$$

is a quotient of polynomials in the unit forward operator $z$. With reference to the estimated parameters of model $\mathcal{M}_1(2, 10, 6)$, the frequency response of $T(z)$ is in good agreement with the Bode plot of (6)(see Fig. 4). In Figs 3(b) and 8, the step response of $T(z)$ is compared with the heat exchanger step response. Save for large variations in velocity, the difference between the corresponding diagrams is small enough.

**Remark 3**

From Appendix 2, it can be seen that parameters $b_2, b_3, \ldots, b_8$ of model $\mathcal{M}_1(2, 10, 6)$ are close to each other. This agrees with the "ramp" behaviour of the first points of the model step response relating $y$ to $u$ (see Figs 3(b) and 8), and can be interpreted as follows. Suppose that the heat exchanger is at the nominal steady state

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<th>Table I</th>
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<tr>
<td>Identification by ELS</td>
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<tr>
<td>$n_a$</td>
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at time $t = 0$ and that a step change $\delta v$ is fed into the system: $u(t) = \bar{v} + \delta v, t > 0$. If $\delta v \ll \bar{v}$, the liquid crossing time is about 8 s, same under nominal conditions. Think of the heat exchanger tube as being ideally consisting of eight parts of the same length and consider the liquid temperatures $\theta_1(L - k L/8, t), k = 0, 1, \ldots, 7$, at the outlet of each part.

The sampling interval being $\Delta = 1$ s, the outlet liquid temperature of the heat exchanger, $\tilde{\theta}_{Io}(t)$, is given by

$$\tilde{\theta}_{Io}(t) = \theta_1\left(L - k \frac{L}{8}, t\right) + \delta \theta_k,$$

where $\delta \theta_k$ is the increment of liquid temperature due to the quantity of heat exchanged along the last $k$ segments of the tube. Such a quantity is presumably equal to $k$ times the heat quantity exchanged along a single segment. This leads to the conclusion that $\delta \theta_k$ should approximately be a linear function of $k$, for $k < 8$.

This observation suggests that the constraint $b_2 = b_3 = \cdots = b_8$ could be imposed at the procedure. Such a constraint, which can be easily incorporated into the ELS algorithm, leads to a model with 12 free parameters instead of the 18 that have to be estimated in $\mathcal{M}_1(2, 10, 6)$. Such a model is discussed in (32). Additional simplifications can be introduced to further reduce the model complexity, see (33).

### Table II
Identification by ELS; the value of $(-U - 13)10^3$ is given

<table>
<thead>
<tr>
<th>$n_a$</th>
<th>10</th>
<th>9</th>
<th>8</th>
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### Table III
Identification by ELS; the value of $(-2AIC - 27)$ is given

<table>
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TABLE IV
Identification by ELS: the value of $r_{0.05}$ is given

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VII. Generalized Minimum Variance Control—an Overview

The minimum variance approach to the control problem is an optimal stochastic technique aimed at minimizing the following performance index:

$$J = E(y(k) - y^o(k))^2,$$

where $y^o$ is the reference signal. A classical introduction to this approach, which goes back to the late sixties, can be found in (34). An important generalization of (11) has been suggested in (15), where the performance index is given by the variance of the so-called generalized error. More precisely, let $P(z^{-1})$ and $Q(z^{-1})$ be two stable transfer functions and define the generalized error $v$ by

$$v(k) = P(z - ) y(k) - y^o(k) + Q(z - ) u(k).$$

The generalized minimum variance control approach seeks to minimize the performance index

$$J = Ev(k)^2.$$  

Remark 4

Performance index (12) includes a number of important special cases, which have been extensively studied in the literature (35). In particular, if $Q(z^{-1}) = 0$ and $P(z^{-1})$ is invertible, the best one can hope for is obviously that

$$y(k) = P(z^{-1})^{-1} y^o(k).$$

This case is referred to as the model reference approach since $P(z^{-1})^{-1}$ expresses the desired behaviour of the control system.

Another typical choice is $P(z^{-1}) = 1$. In such a case, assuming $Q(z^{-1})$ is invertible, the best one can hope for is that

$$u(k) = Q(z^{-1})^{-1}(y(k) - y^o(k)).$$

Consequently, $Q(z^{-1})^{-1}$ can be interpreted as the controller transfer function suggested by the optimization criterion adopted.

This latter approach is referred to in the literature as the detuned model reference...
Sergio Bittanti, Fabio Romeo and Riccardo Scattolini

approach, see e.g. (20). For more discussion on generalized minimum variance control, the reader is referred to the survey papers (36) and (37). In these papers, a more general performance index is adopted, by letting

\[ v(k) = P(z^{-1})y(k) - R(z^{-1})y^o(k) + Q(z^{-1})u(k) \]

where \( R(z^{-1}) \) is the transfer function of a reference signal prefilter. However, the generalization to the case \( R(z^{-1}) \neq 1 \) being of minor interest here, we shall concentrate on the case \( R(z^{-1}) = 1 \). In the linear case, the generalized minimum variance control law is given in Proposition 1 below, whose proof can be found in (15).

**Proposition 1.** Consider system (7) and performance index (12). Let \( P_N(z^{-1}), P_D(z^{-1}), Q_N(z^{-1}) \) and \( Q_D(z^{-1}) \) be the numerators and denominators of \( P(z^{-1}) \) and \( Q(z^{-1}) \) respectively. Define

\[
\begin{align*}
F(z^{-1}) &= Q_D(z^{-1})\tilde{F}(z^{-1}) \\
G(z^{-1}) &= P_D(z^{-1})\{B(z^{-1})E(z^{-1})Q_D(z^{-1}) + C(z^{-1})Q_N(z^{-1})\} \\
H(z^{-1}) &= P_D(z^{-1})Q_D(z^{-1})C(z^{-1}),
\end{align*}
\]

where polynomials \( E(z^{-1}) \) and \( \tilde{F}(z^{-1}) \) satisfy:

\[
C(z^{-1})P_N(z^{-1}) = A(z^{-1})P_N(z^{-1})E(z^{-1}) + z^{-d}\tilde{F}(z^{-1}).
\]

Assume the reference signal \( y^o \) to be independent of the noise and also that

\[
A(z^{-1}) = C(z^{-1})V(z^{-1}),
\]

is an asymptotically stable polynomial. Then, the minimum variance control law is given by

\[
F(z^{-1})y(k) + G(z^{-1})u(k) - H(z^{-1})y^o(k + d | k) = 0,
\]

where \( y^o \) is the optimal \( d \)-steps ahead prediction of \( y^o \).

**Remark 5**

If no a priori information on \( y^o \) is available, then typically \( y^o(k) \) is used instead of \( y^o(k + d | k) \) in (18). This corresponds to assuming that

\[
v(k) = P(z^{-1})y(k + d) - y^o(k) + Q(z^{-1})u(k)
\]

instead of (12a). Then (19) leads to

\[
F(z^{-1})y(k) + G(z^{-1})u(k) - H(z^{-1})y^o(k) = 0
\]

as an optimal control law. System (7) and control law (20) can be represented by the block diagram in Fig. 9.

**Remark 6**

Notice that (17) is the characteristic polynomial of the system of Fig. 9. This polynomial can be rewritten as follows:

\[
\Delta(z^{-1}) = C(z^{-1})V(z^{-1}),
\]
where

$$V(z^{-1}) = A(z^{-1})Q_N(z^{-1})P_D(z^{-1}) + B(z^{-1})Q_D(z^{-1})P_N(z^{-1}).$$

(22)

From (21), it follows that the zeros of $z^nC(z^{-1})$ are poles of the closed-loop system. Whenever the model residual process, $C(z^{-1})e(k)$, is stationary, these poles are inside the unit circle. This follows from the spectral factorization theorem (34). As for the zeros of $V(z^{-1})$ in the model reference approach, it is apparent from (22) that they are those of $B(z^{-1})$ plus those of $P_N(z^{-1})$. Therefore, this approach is not suitable for controlling nonminimum phase processes. No zero cancellation occurs in the detuned model reference case, and (22) becomes

$$V(z^{-1}) = A(z^{-1})Q_N(z^{-1}) + B(z^{-1})Q_D(z^{-1}).$$

(23)

$Q(z^{-1})$ should be designed so as to secure that the closed loop poles determined by polynomial (23) are conveniently located in the complex plane.

**VIII. Detuned Model Reference Control of the Heat Exchanger**

In full agreement with the magnitude ratio diagram of (6) (Fig. 4), the zeros of polynomial $z^nB(z^{-1})$ of the estimated model $M_1(2,10,6)$ are close to the unit circle.
Therefore, in view of Remark 6, the model reference approach is not suited to deal with the control problem at hand (unless some ad hoc trick is adopted). The interested reader is referred to (38) where the bad control action achieved by means of this approach is exposed. We shall therefore concentrate here on the detuned approach, where \( P(z^{-1}) = 1 \). Three typical choices of \( Q(z^{-1}) \) are discussed in order.

(i) \( Q(z^{-1}) = h, \) a constant

This case is named extended minimum variance control in (39). A theoretical analysis of the corresponding control system is discussed in (40). With such a choice of \( Q(z^{-1}) \), (23) becomes

\[
\mathcal{V}(z^{-1}) = hA(z^{-1}) + B(z^{-1}).
\]

Denoting the degree of polynomial \( \mathcal{V}(z^{-1}) \) by \( n_v \), the solutions of the characteristic equation

\[
z^{nv}\mathcal{V}(z^{-1}) = 0 \quad (24)
\]

coincide with the zeros of \( z^{nv}B(z^{-1}) \) for \( h = 0 \), and they approach the zeros of \( z^{nv}A(z^{-1}) \) when \( |h| \to \infty \). From the interpretation of \( Q(z^{-1}) \) given in Remark 4 (and the model gain being negative), it follows that only negative values of \( h \) are of interest.

The root locus of (24) defined by the solutions parametrized by \( h, h < 0 \), is given in Fig. 10(a). This locus suggests that \( h = -18 \) is a reasonable choice. The corresponding controller, obtained by (13)–(18), is given as \( \mathcal{Q}_4 \) in Appendix 4. This controller, as well as the ones subsequently derived, has been tested on the heat exchanger by subjecting the digital control system of Fig. 11 to the following series of stepchanges in the principal variables:

\[
y^c(k) = \begin{cases} 0, & k < 0 \\ -0.672°C, & k \geq 0 \end{cases}; \quad \tilde{\theta}_{lb}(t) = \begin{cases} \tilde{\theta}_{lb}, & t < t_1 \\ 66.5°C, & t \geq t_1 \end{cases}; \\
\theta_{cl}(t) = \begin{cases} \theta_{cl}, & t < t_2 \\ 117°C, & t \geq t_2 \end{cases}; \quad t_1 = 45 \text{ s}, t_2 = 90 \text{ s}.
\]
FIG. 10. Root locus of (24) for $Q(z^{-1}) = h$ and $Q(z^{-1}) = h(1-z^{-1})$ [(a) and (b), respectively].
This test will be denoted as the *three steps test*. From the result of this test relative to controller $C_1$ (Fig. 12(a)), it can be seen that the transients are smooth. However, the output does not coincide with $y^o = 98°C$ at the steady state. This offset is not surprising. Indeed, the transfer function from $y^o$ to $y$ of the system represented in Fig. 9 turns out to be given by

$$S(z^{-1}) = \frac{z^{-d}}{1 + Q(z^{-1})A(z^{-1})B(z^{-1})} \quad (25)$$

In terms of model $\mathcal{M}_1(2, 10, 6)$ and controller $C_1$, the gain of this transfer function is given by

$$S(1) = \frac{1}{1 + h \frac{A(1)}{B(1)}} = 0.548.$$ 

This value is in good agreement with the experimental gain of 0.537 deduced from Fig. 12(a).

From (25), it is apparent that $(1 - z^{-1})$ should divide polynomial $Q(z^{-1})A(z^{-1})$ in order to obtain a unit gain for $S(z^{-1})$. This observation leads to the following two designs.

(ii) $Q(z^{-1}) = h(1 - z^{-1})$

The corresponding root locus defined by the solutions of (24) as $h$ varies from 0 to $-\infty$ is given in Fig. 10(b). Controller $C_2$ of Appendix 4 is derived from (13)-(18) by taking $h = -112$. The results of the corresponding three steps test are given in Fig. 12(b). A stepchange in the set point does not give rise to any offset. However, the disturbances to the inlet fluid and steam temperatures do produce offsets of the outlet liquid temperature. This is in agreement with the fact that the gain of the transfer function from the disturbance $e$ to the output $y$ of the control system of Fig. 9 may differ from 0 even if $(1 - z^{-1})$ divides $Q(z^{-1})$. In other words, including factor $(1 - z^{-1})$ into polynomial $Q(z^{-1})$ is not equivalent to introducing an integrator in the closed-loop.

(iii) $Q(z^{-1}) = h$, with integral action

The following simple algorithm can be applied in order to impose a controller integral action. Let $\mathcal{M}_1(3, 10, 6)$ be the ARMAX model deduced from the estimated
model $\mathcal{M}_1(2,10,6)$ by replacing polynomial $A(z^{-1})$ by $(1 - z^{-1}) A(z^{-1})$. Let $\mathcal{G}$ be a controller of the type

$$\tilde{F}(z^{-1})y(k) + G(z^{-1})u(k) - \tilde{H}(z^{-1})y^\circ(k) = 0$$

suitable for controlling $\mathcal{M}_1(3,10,6)$. Then, a suitable integral action controller for $\mathcal{M}_1(2,10,6)$ is easily obtained by means of (20), i.e.

$$F(z^{-1}) = F(z^{-1})$$

(26a)
\[ G(z^{-1}) = (1 - z^{-1}) G(z^{-1}) \] (26b)
\[ H(z^{-1}) = \bar{H}(z^{-1}). \] (26c)

As for the design of \( Q \), it is accomplished by the detuned model reference approach with \( Q(z^{-1}) = h \), whose root locus is again the one of Fig. 10(b). Assuming, as in (ii), \( h = -112 \) and applying (26), controller \( C_3 \) of Appendix 4 is finally obtained. The three steps response is then given in Fig. 12(c).

A number of further controllers are discussed in (33) and (38) where successful applications of the pole-placement approach, (41), are also reported.
FIG. 13. Self-tuning control of the heat-exchanger by controllers $\mathcal{C}_d(2,2,0)$, $d = 1$ (a), $d = 2$ (b), $d = 3$ (c), $d = 4$ (d), $d = 5$ (e), $d = 6$ (f). The diagrams of $v$ (dotted lines) and of $\theta_{\text{io}}$ (continuous lines) are given. The scales are the same as Fig. 12.

IX. A Self-tuning Control Algorithm

The idea of a system capable of controlling itself is at the root of adaptive control. A number of adaptive control schemes and successful applications have been reported in the literature, see e.g. (2, 39, 42–45). However, there are many theoretical and practical problems open to further research. In this paper, attention is focused on a simple self-tuning version of the detuned model reference approach introduced in Section VI. More precisely, let

$$\hat{y}(k) = \frac{1}{P_d(z^{-1})} y(k)$$

(27)
Sergio Bittanti, Fabio Romeo and Riccardo Scattolini

\[ \ddot{y}(k) = y^o(k) - Q(z^{-1})u(k) \]  
\[ \tilde{G}(z^{-1}) = B(z^{-1})E(z^{-1}) \]  
\[ \tilde{H}(z^{-1}) = C(z^{-1}) \] 

where both \( E(z^{-1}) \) and \( \tilde{F}(z^{-1}) \) satisfy (16). It can be shown (46) that the generalized minimum variance optimal control law (20) can be reformulated as follows:

\[ F(z^{-1})\ddot{y}(k) + G(z^{-1})u(k) - \tilde{H}(z^{-1})y^o(k) = 0. \]  

Furthermore, letting

\[ \Psi(k) = P(z^{-1})y(k) \]  

it can be seen that

\[ \tilde{H}(z^{-1})\Psi(k) = F(z^{-1})\ddot{y}(k-d) + G(z^{-1})u(k-d) + \xi(k), \]

where \( \xi(k) \) is given by

\[ \xi(k) = C(z^{-1})E(z^{-1})e(k). \]

In adaptive control, polynomials \( \tilde{F}(z^{-1}), \tilde{G}(z^{-1}) \) and \( \tilde{H}(z^{-1}) \) or \( F(z^{-1}), G(z^{-1}) \) and \( H(z^{-1}) \) must be estimated in real time. Basically, this estimation can be carried out in two ways. The first approach consists of estimating the model polynomials \( A(z^{-1}), B(z^{-1}) \) and \( C(z^{-1}) \) and then deriving the controller polynomials \( \tilde{F}(z^{-1}), \tilde{G}(z^{-1}) \) and \( \tilde{H}(z^{-1}) \) or \( F(z^{-1}), G(z^{-1}) \) and \( H(z^{-1}) \) by means of (16), (29) and (30) or (13)-(16). This is the so-called explicit approach. On the contrary the second approach requires the feedback parameters to be estimated directly from the data, without previously estimating the model parameters. In fact, once \( P(z^{-1}) \) is given, variables \( \ddot{y} \) and \( \Psi \) defined by (27) and (32) can be updated in real time. Equation (33) is then a linear stochastic model having \( \ddot{y} \) and \( y \) as inputs and \( \Psi \) as output. The parameters of such a model can be estimated by an ELS-type algorithm. The parameters thus obtained are then fed into (31) to derive the optimal control law.

This implicit self-tuning algorithm will be applied to the heat exchanger control problem in the proceeding section. The controller resulting will be denoted by \( G_r(n_F, n_G, n_H) \), where \( n_F, n_G \) and \( n_H \) are the degrees of polynomials \( \tilde{F}(z^{-1}), \tilde{G}(z^{-1}) \) and \( \tilde{H}(z^{-1}) \) respectively.

Remark 7

The advantage of considering the control law in the form of (31) instead of (20) is that, contrary to \( F(z^{-1}), G(z^{-1}) \) and \( H(z^{-1}) \), polynomials \( \tilde{F}(z^{-1}), \tilde{G}(z^{-1}) \) and \( \tilde{H}(z^{-1}) \) do not explicitly depend upon \( P(z^{-1}) \) and \( Q(z^{-1}) \) [compare (13)-(16) with (16), (29), (30)]. This was obtained due to having defined \( \ddot{y} \) and \( y^o \) in (27) and (28).

Remark 8

If \( n_e = 0 \), it follows from (8) that \( C(z^{-1}) \) reduces to 1. Thus (33) becomes a classical regression of \( \Psi \) on \( \ddot{y} \) and \( u \), whose parameters can be simply estimated by the RLS algorithm.
Remark 9
An effective way to overcome the steady state offset problem is to include a dc term in (33). As is well known, this can be achieved by a slack input variable, whose value is fixed and equal to 1. See [(39), Section 2.4] for this point.

X. Self-tuning Control of the Heat Exchanger

As in Section VIII, attention is concentrated on the detuned model reference approach. More precisely, the results which will be discussed in this section refer to the case where \( P(z^{-1}) = 1 \) and

\[
Q(z^{-1}) = h(1 - z^{-1}), \quad h = -400.
\]

Further choices are discussed in (47). Figure 13 shows the results of the three steps test for the implicit self-tuning control system of the previous section, with controllers \( e_{d}(2,2,0) \), \( d = 1,2,\ldots,6 \).

In all these cases, the parameters of (33) together with the dc term have been estimated by the RLS algorithm with a forgetting factor equal to 0.99. It can be seen from Fig. 13 that \( d = 4 \) leads to the best performance; \( d = 3 \) is also acceptable, whereas \( d = 1 \) or \( d = 2 \) gives rise to unstable systems. Slower and slower control systems are obtained if \( d \) is increased beyond 4.

Remark 10
From Figs 12 and 13, it can be concluded that the self-tuning controller with \( d = 4 \) exhibits an acceptable performance with five parameters only. On the contrary, controllers \( e_{1}-e_{3} \) used in Section VIII are defined by about 20 parameters. Moreover, the time necessary to identify model \( \mathcal{H}(2,10,6) \), by means of which controllers \( e_{1}-e_{3} \) were designed, was much longer than the time necessary to achieve an acceptable response when using the self-tuning scheme.

XI. Conclusions

In this paper, the control problem of a liquid-saturated steam heat exchanger was examined by various stochastic digital control techniques, both adaptive and nonadaptive. In all cases, the nonadaptive controller designs were based on an identified ARMAX model.

The main conclusions we can draw from this case study can be summarized as follows:

(1) An identification procedure based on a simple estimation algorithm and on commonly used model complexity selection criteria was applied. Subject to a sensible planning of the identification experiment, it led to a good ARMAX model.

(2) Due to a suitable controller design methodology, control systems having good performance can be designed using the identified process model. However, this model being rather complex, the controllers so derived are characterized by many parameters.

(3) An implicit self-tuning algorithm was successfully applied to the control of the
Sergio Bittanti, Fabio Romeo and Riccardo Scattolini

heat exchanger. An acceptable performance was secured by a simple controller with few parameters. The choice of the model delay plays a major role in achieving these results.

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References

Appendix 1. Process Data

Nominal values

\[ \bar{\theta}_u = 65^\circ C, \quad \bar{\theta}_w = 98.672^\circ C, \quad \bar{\theta}_s = 120^\circ C, \quad \bar{\ddot{\theta}}_i = 0.3 \text{ ms}^{-1}, \quad \ddot{\theta}_i = 754 \text{ J m}^{-2} \text{ s}^{-1} \text{ C}^{-1}. \]

Data

\[ L = 2.44 \text{ m}, \quad D_1 = 0.0547 \text{ m}, \quad D_2 = 0.0613 \text{ m}, \quad v = 0.8, \quad x_1 = 3510 \text{ J m}^{-2} \text{ s}^{-1} \text{ C}^{-1}, \quad c_1 = 4186 \text{ J kg}^{-1} \text{ C}^{-1}, \quad c_2 = 393.5 \text{ J kg}^{-1} \text{ C}^{-1}, \quad \mu_1 = 0.223 \text{ kg m}^{-1}, \quad \mu_2 = 0.532 \text{ kg m}^{-1}, \quad v_m = 0.1 \text{ ms}^{-1}, \quad v_M = 0.5 \text{ ms}^{-1}, \quad \gamma = -1 \text{ s}^{-1}, \quad \delta = 1 \text{ s}^{-1}. \]

Correspondingly, \( \tau_1 = 7.2 \text{ s}; \quad \tau_2 = 0.31 \text{ s}; \quad \bar{\beta} = 0.1913. \]

These data are derived from [18] p. 225.

Appendix 2. The ELS Identification Algorithm

Consider the class of equation error models, given by (10). The least-squares criterion of fit over the interval \([1, k]\) is given by

\[ J_k = \sum_{i=1}^{k} \rho(k)^{k-i} \epsilon(i)^2, \]  

(A2.1)
where e(i) is the model prediction error at time i and \( \rho(k)^{k-i} \) is the relative weight given to e(i).

In the classical LS criterion, it is assumed that \( \rho \) is a constant, e.g. \( \rho(k) = 1 \), for every i. More generally, we assume that \( \rho(k) \in (0, 1) \), for every k.

As the prediction error of model (10) is linear in the parameters, the minimum point of \( J_k \) can be given a closed form. Its (approximate) recursive expression can be written in a variety of ways. Letting

\[
\phi(k) = [y(k-1)y(k-2) \ldots y(k-n_u)u(k-d)u(k-d-1) \ldots u(k-d-n_b+1)]'
\]

the RLS is, e.g., given by

\[
\hat{\xi}_k = \hat{\xi}_{k-1} + \left[ \rho(k) + \phi(k)' V(k-1) \phi(k) \right]^{-1} V(k-1) \phi(k) [y(k) - \phi(k)' \hat{\xi}_{k-1}]
\]

(A2.2)

\[
V(k) = \left\{ V(k-1) - \left[ \rho(k) + \phi(k)' V(k-1) \phi(k) \right]^{-1} V(k-1) \phi(k) \phi(k)' V(k-1) \right\} \frac{1}{\rho(k)}
\]

(A2.3)

where \( \hat{\xi}_k \in \mathbb{R}^{n+\nu} \) is the parameter estimate at time k, namely the current minimizer of (A2.1); and \( V(k) \in \mathbb{R}^{(n_u+n_b)\times(n_u+n_b)} \). For simplicity, arbitrary initializations of the type \( \hat{\xi}_0 = 0 \), \( V(0) = \alpha I, \alpha > 0 \), are usually considered. This can be compensated for by letting \( \rho(.) \) be monotonically increasing from an initial value \( \rho(0) \) to 1. In such a way, the first estimates of \( \hat{\xi}_k \) which are mostly affected by the initialization, are discarded more quickly than in classical RLS. In other words, \( \rho \) plays the role of a data forgetting factor (6, 48). In the trials reported in this paper, \( \rho \) was determined as follows:

\[
\rho(k) = \hat{\rho} \rho(k-1) + (1 - \hat{\rho}), \quad \hat{\rho} = 0.99, \quad \rho(0) = 0.9.
\]

Both the LS and RLS algorithms are simple and easy to implement. However, they produce biased estimates when \( n_u \neq 1 \), as in the general ARMAX model (7). A number of variants have been proposed in the literature in order to retain algorithm simplicity while overcoming this problem. One of the simplest is the extended least-squares (ELS) algorithm, which is again defined by (A2.2) and (A2.3) with \( \hat{\xi}_k \in \mathbb{R}^{n_a+n_b+n_c} \), \( V(k) \in \mathbb{R}^{(n_a+n_b+n_c)\times(n_a+n_b+n_c)} \) and

\[
\phi(k) = [y(k-1)y(k-2) \ldots y(k-n_a)u(k-d)u(k-d-1) \ldots u(k-d-n_b+1)\omega(k-1)\omega(k-2) \ldots \omega(k-n_c)]'
\]

(A2.4)

Here, \( \omega \) is a slack variable which is recursively defined by

\[
\omega(k) = y(k) - \phi(k)' \hat{\xi}_{k-1}.
\]

(A2.5)

Notice that (9), (A2.4) and (A2.5) entail that

\[
\hat{C}_{k-1}(z) \omega(k) = f(k),
\]

(A2.6)

where \( \hat{C}_{k-1}(z^{-1}) \) is polynomial (8) with parameters \( c_i \) substituted by their estimates at time \( k-1 \) and \( f(k) \) is a function of past values of \( y \) and \( u \). From (A2.6) it is apparent that the stability of \( z^{n_a} \hat{C}_{k-1}(z^{-1}) \) is necessary to guarantee the nondivergence of \( \omega \). Therefore, at each k, the stability of estimated polynomial \( z^{n_a} \hat{C}_{k-1}(z^{-1}) \) was tested. In case of instability, the spectrum \( S(z) = \hat{C}_{k-1}(z) \hat{C}_{k-1}(z^{-1}) \) was computed. Then, the stable polynomial \( \hat{C}_{k-1}(z) \) such that \( S(z) = \hat{C}_{k-1}(z) \hat{C}_{k-1}(z^{-1}) \) was found. Such a polynomial does exist in view of the spectral factorization theorem (34) and can be found by the Bauer algorithm (49).

Then, \( \hat{C}_{k-1}(z^{-1}) \) was used in (A2.6) in place of the instable polynomial \( \hat{C}_{k-1}(z^{-1}) \).

For more details on the algorithm and the analysis of its asymptotic properties we refer the interested reader to (50–54).
Appendix 3. Model $\mathcal{M}_1(2, 10, 6)$

Model $\mathcal{M}_1(2, 10, 6)$ is given by

\[
(1 - 0.423 z^{-1} + 0.022 z^{-2})y(k) = (-0.887 - 1.619 z^{-1} - 1.606 z^{-2} - 1.608 z^{-3} - 1.608 z^{-4} - 1.607 z^{-5} - 1.608 z^{-6} - 1.572 z^{-7} - 0.908 z^{-8} - 0.095 z^{-9})u(k-1) + (1 + 0.651 z^{-1} + 0.231 z^{-2} + 0.189 z^{-3} + 0.164 z^{-4} + 0.101 z^{-5} + 0.063 z^{-6})e(k)
\]

where $u$ is in $\text{ms}^{-1}$ and $y$ in °C.

Model $\mathcal{M}_1(2, 10, 6)$ characteristics

- Prediction error variance ($\mu^2$): $0.111 \times 10^{-5}$
- Rissanen criterion $((-U-13)10^3)$: 671
- Akaike criterion $((-2AIC-27)10^3)$: 403
- $A(1) = 0.5993$
- $B(1) = -13.1167$
- $C(1) = 2.4003$

Model $\mathcal{M}_1(2, 10, 6)$ singularities

- Zeros of $z^n A(z^{-1})$:
  - Absolute value: 0.062, 0.361
  - Phase: 0, 0

- Zeros of $z^n B(z^{-1})$:
  - Absolute value: 0.994, 0.994, 0.979, 0.979, 0.960, 0.960, 0.969, 0.969, 0.969, 0.994, 0.994, 0.979, 0.979, 0.960, 0.960, 0.969, 0.969, 0.969, 0.130
  - Phase: 44, -44, 88, -88, 132, -132, 169, -169, 180

- Zeros of $z^n C(z^{-1})$:
  - Absolute value: 0.654, 0.654, 0.667, 0.667, 0.574, 0.574
  - Phase: 48, -48, 157, -157, 105, -105
Appendix 4. Controllers Designed by the Various Techniques

$\mathcal{C}_1$:

\[ F_1(z^{-1}) = 1.0740 + 0.20899 z^{-1} - 0.18952 z^{-2} + 0.16423 z^{-3} + 0.10087 z^{-4} + 10^{-1} 0.63370 z^{-5} \]

\[ G_1(z^{-1}) = -10^2 0.18923 - 10^2 0.13359 z^{-1} - 5.7784 z^{-2} - 5.0262 z^{-3} - 4.5702 z^{-4} - 3.4260 z^{-5} - 2.7511 z^{-6} - 1.5721 z^{-7} - 0.90766 z^{-8} - 10^{-1} 0.94760 z^{-9} \]

\[ H_1(z^{-1}) = C(z^{-1}). \]

$\mathcal{C}_2$:

\[ F_2(z^{-1}) = F_1(z^{-1}) \]

\[ G_2(z^{-1}) = -113.62 + 37.729 z^{-1} + 45.693 z^{-2} + 3.10736 z^{-3} + 1.2427 z^{-4} + 5.5357 z^{-5} + 2.6191 z^{-6} + 5.5714 z^{-7} - 0.90766 z^{-8} - 0.9476 z^{-9} \]

\[ H_2(z^{-1}) = C(z^{-1}). \]

$\mathcal{C}_3$:

\[ F_3(z^{-1}) = 2.0740 - 0.2140 z^{-1} + 0.2119 z^{-2} + 0.16423 z^{-3} + 0.1009 z^{-4} + 0.0634 z^{-5} \]

\[ G_3(z^{-1}) = -113.61 + 38.617 z^{-1} + 47.312 z^{-2} + 4.7131 z^{-3} + 2.9508 z^{-4} + 7.1438 z^{-5} + 4.2258 z^{-6} + 7.1795 z^{-7} + 0.6644 z^{-8} + 0.8129 z^{-9} + 0.0948 z^{-10} \]

\[ H_3(z^{-1}) = C(z^{-1}). \]